Geometric Modeling of Dubins Airplane Movement and its Metric

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ABSTRACT

The time-optimal trajectory for an airplane from some starting point to some final point is studied by many authors. Here, we consider the extension of robot planer motion of Dubins model in three dimensional spaces. In this model, the system has independent bounded control over both the altitude velocity and the turning rate of airplane movement in a non-obstacle space. Here, in this paper a geometrization of time-optimal trajectory of Dubins airplane is also obtained. More intuitively, the metric related to this phenomenon is described as a geometry in space. It is shown that the distance traveled in movement of an airplane obeys certain conditions of a well-known geometry called Finsler geometry. Moreover, it is proved that the geometry of movement of an airplane is a special Finsler metric known as Randers metric, and therefore, time-optimal paths are geodesics of Randers metric.

KEYWORDS

Dubins airplane, Finsler metric, Indicatrix, Randers Metric, Control, Time-Optimal trajectories

1. INTRODUCTION

One of the present authors in his previous works has studied geometry of time-optimal trajectories of an object pursuing a moving target, without any bounded control in a non-obstacle space, cf. [1]. Here we study geometry of a system which has independent bounded control over the altitude velocity as well as the turning rate of movement in a non-obstacle space. More intuitively, this phenomenon is described as a geometry in space.

1.1 Geometrization of a Mathematical Model

Euclidean geometry known as one of the oldest sciences, contributes greatly in the mathematical modeling of engineering problems. We may link a geometry to an optimization problem as follows.

In the general sense, Optimization refers to choosing the best element from some set of available alternatives. If elements of this set are any kind of paths, we may correspond a metric to this set. This metric together with the space of all possible paths is called geometry of the model.

In this case, the optimized trajectories are geodesics of the prescribed metric. We will refer to this observation as geometrization of a mathematical model.

Once geometry of a movement is defined we will be able to use the deep notions of differential geometry to determine the optimized paths of the movement.

It is noteworthy to remark that, despite of its appearance, these two phrases "Geometrization of a Model" and "Geometric Model" are quite different. In fact, the former fraise is described before, but the later one which is used frequently in optimization problems, refers to application of geometric laws to explain a mathematical model.

1.2 Dubins Airplane Model

Finding the best time-optimal trajectories for a simple Robot model has been first done by Dubins in 1957. Using a geometrical method, he proved that the shortest path for a car-like differential robot that can only move forward with a constant speed and its rotation radius to the left and right is bounded, are a mixture of circles with constant radius and straight lines. In 1990, this model was developed by Reeds and Shepp for a robot which is able to move backward and forward.

In 1994, Reed and Shepp works was reviewed again, and using the method of finding the best controls, time-optimal trajectories for Reeds and Shepp's car, that moves in a non-obstacle space, was presented. In 1995, a method for designing the shortest paths, in a position that there is

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an obstacle on the surface of the motion, but with considering the limitation in the curvature of the motion's path was studied. Also in 1996, the paths of the shortest time of passage for a differential robot purchasing a moving target has been studied cf. [4].

Other efforts, in relation to the time-optimal trajectories and the paths with the minimum number of rotation of the wheels for a differential robot have also been made cf. [5],[6] and [7].

In 2008, by extending Dubins method for an airplane moving in space without any obstacle heading for a specified target, time optimal paths is studied cf. [11].

Also in 2009, Hota and Ghose introduced a Dubins model based strategy to determine the optimal path of a Miniature Air Vehicle, constrained by a bounded turning rate, that would enable it to fly along a given straight line, starting from an arbitrary initial position and orientation. They used a modification of Dubins path method to obtain the complete optimal solution to the problem in all its generality cf. [13].

Here, in this paper we use their methods to describe geometry of these optimal paths, and we show that the geometry of Dubins airplane Model is a special case of Finsler geometry called Randers geometry. Therefore, time optimal paths of the system are geodesics of a Randers space.

2. Preliminaries concepts of Finsler geometry

To measure the length of a smooth curve C parameterized by a map $c = c(t), a \le t \le b$, in a manifold M, it suffices to define a nonnegative scalar function $F(x,.)$ on every tangent space T_xM. Then the length of C is defined by,

$$L_F(C) = \int_a^b F(c(t), c'(t))dt. \tag{1}$$

It is required that $L_F(C)$ be independent of parameterization. F must be positively homogeneous with degree one,

$$F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0. \tag{2}$$

Let $M = \mathbb{R}^n$ be an n-dimensional C^∞ Euclidean space. Denote by T_xM the set of all tangent vectors at the point $x \in M$, called tangent space, and by $TM := \cup_{x \in M} T_xM$ the set of all tangent spaces at $x \in M$ called tangent bundle. Each element of TM has the form (x, y) consisting of the point $x \in M$ and the tangent vector $y \in T_xM$ at the point x called respectively state (position) and direction of movement. If $y \neq 0$ then the tangent bundle is denoted by TM_0.

The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. A vector field Y on M is a map $Y : M \to TM$ with the property

$$\pi o Y = Id_M.$$

Definition 2.1. A Finsler structure on M is a function,

$$F : TM \longrightarrow [0, \infty)$$

with the following properties:

(i) Regularity condition: F is C^∞ on the slit tangent bundle TM_0.

(ii) Positive homogeneity condition: for all $\lambda > 0$,
$$F(x, \lambda y) = \lambda F(x, y).$$

(iii) Strong convexity: The $n \times n$ Hessian matrix
$$(g_{ij}) := \frac{\partial^2 F^2}{2\partial y^i \partial y^j},$$
is positive-definite at every point of TM_0.

The pair (M, F) is called a Finsler space.

According to the Fundamental Inequality theorem cf. [2] page 7, one can show that a Finsler structure defines a norm on the tangent space T_xM called Minkowski norm.

In the original sense, a geodesic is a generalization of the notion of a "straight line" in Euclidean space. In the presence of a metric, rather than Euclidean metric, in the space, geodesics are no more straight lines.

More precisely, a piecewise C^∞ curve $\sigma : [a, b] \to M$ with $\sigma(a) = p$ and $\sigma(b) = q$ on the space M with the metric F is said to be a *geodesic* if it is a minimal curve and if it has a constant velocity. Hence, geodesics are known to be (locally) the shortest path between points in the space.

Let $g := g_{ij}$ be a Finslerian metric on M, then it gives rise to an inner product $g(.,.) := <.,.>$ on the tangent space T_xM. In the local coordinate system $\{U, (x^i)\}$, $\forall y, z \in T_xM$ we have $g(y, z) = g_{ij} y^i z^j$.

Each inner product defines a norm for a vector $y \in T_xM$ with respect to the metric g, $\| y \| := <.,.>_g$.

Hence a vector on the tangent space $y \in T_xM$, can have different kinds of norms, related to the different kinds of the metric g defined on M. If the norm $\| . \|$ on the tangent space T_xM is related to a mathematical model, then the metric g on M determines the geometry of the model and shortest paths are geodesics of g. This metric together with the space of all possible paths is called geometry of the prescribed mathematical model and this observation is called geometrization of the mathematical model.

2.1. Indicatrix and finding new metric by shifting it

Given a Minkowski space (V, F), let

$$S_F := \{y \in v : F(y) = 1\}. \quad (3)$$

S_F is a closed hypersurface around the origin, Which is diffeomorphic to the standard sphere $S^{n-1} \subset \mathbb{R}^n$. S_F is called the indicatrix of F. Now we are going to construct Minkowski norms by shifting the indicatrix of a Minkowski norm. Let (V, Φ) be a Minkowski space and let $\upsilon \in V$ with $\Phi(-\upsilon) < 1$. Then the shifted set, $S_\Phi + \{\upsilon\}$, contains the origin of V .We can define a function $F : V \to [0, \infty)$ as follows: for any $y \in V - \{0\}, F(y)$, is the unique positive number $t > 0$ such that

$$\frac{y}{t} \in S_\Phi + \{\upsilon\}, \quad (4)$$

It is easy to see that F has the following properties:

(a) $F > 0$ for any $y \in V - \{0\}$.

(b) $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$.

(c) $S_F = S_\Phi + \{V\}$.

For any $y \in V - \{0\}, F(y)$ can be determined by the following equation, cf. [3].

$$F(y) = \Phi(y - F(y)\upsilon). \quad (5)$$

F is called the Minkowski norm generated by (Φ, υ). One can easily show that if $F = F(y)$ is generated by (Φ, υ), then $\Phi = \Phi(y)$ is generated by $(F, -\upsilon)$.

2.2. Zermelo Navigation problem

Consider an object moving in a metric space, such as Euclidean space, pushed by an internal force and an external force field. The shortest time problem is to determine a curve from one point to another in the space, along which it takes the least time for the object to travel. This problem in some special cases was studied by E. Zermelo cf. [8], hence called the Zermelo navigation problem.

Here we shall discuss the navigation problem in the most general case. Suppose that an object on a Finsler space (M, Φ) is pushed by an internal force U with constant length, $\Phi(x, u_x) = c$, and while it is pushed by an external force field V with $\Phi(x, -V_x) < c$. The combined force at x is $T_x := U_x + V_x$. The condition, $\Phi(x, -V_x) < c$, guarantees that the object can move forward in any direction. Due to the friction, the object moves on M at a speed proportional to the combined force T. For the sake of simplicity, one may assume that $c = 1$ and the velocity vector at any point $x \in M$ is equal to T_x. Given a pair of points $p, q \in M$, let C be an arbitrary piecewise C^∞ curve in M. Since $\Phi(x, U_x) = 1$, we have,

$$\Phi(x, T_x - V_x) = \Phi(x, U_x) = 1. \quad (6)$$

On the other hand, for any vector $y \in T_x M - \{0\}$, there is a unique solution $F = F(x, y) > 0$ to the following equation,

$$\Phi(x, \frac{y}{F} - V_x) = 1. \quad (7)$$

Observe that for any $\lambda > 0$,

$$1 = \Phi(x, \frac{\lambda y}{\lambda F(x,y)} - V_x) = \Phi(x, \frac{\lambda y}{F(x, \lambda y)} - V_x). \quad (8)$$

By the uniqueness,

$$F(x, \lambda y) = \lambda F(x, y). \quad (9)$$

One can show that $F_x := F|_{T_x M}$ is a Minkowski norm on $T_x M$. Thus $F = F(x, y)$ is a Finsler metric on M. Comparing (6) and (7), one can see that the combined force T_x has unit F-length,

$$F(x, T_x) = 1. \quad (10)$$

This observation leads to following Lemma, cf. [3] page 21.

Lemma 1. Let (M, Φ) be a Finsler space and V be a vector field on M with $\Phi(x, V_x) < 1$, for all $x \in M$. Define $F : TM \to [0, \infty)$ by (7). For any piecewise C^∞ curve C in M, the F-length of C is equal to the time for which the object travels along it.

3. GEOMETRIZATION OF DUBINS AIRPLANE MOTION BY FINSLER GEOMETRY

The control systems consider behavior of a system whose state at any instant of time is characterized by $n_x \geq 1$ real numbers $x_1, ..., x_{n_x}$. The vector space of the system under consideration is called the phase space. It can naturally be assumed that the phase space is an n_x-dimensional smooth manifold. In other words; the phase space, is the space of all states that can happen for the system. The vector $x(t)=(x_1(t),...,x_{n_x}(t)),\ x(t)\in\mathbb{R}^n$ represents the state (or phase) variables. It is assumed that the system can be controlled, i.e., the system is equipped with controllers whose position dictates its future evolution. These controlers are characterized by points $u=(u_1,...,u_{n_u})\in\mathbb{R}^{n_u}, n_u\geq 1$, called the control variables cf. [9].

In the vast majority of optimal control problems, the values that can be assumed by the control variables are restricted to a certain control region U, which may be any set in $\mathbb{R}^{n_u}$.

A nontrivial part of any control problem is modeling the system. The objective is to obtain the simplest mathematical description that adequately predicts the response of the physical system to all admissible controls. We discuss about systems described by the ordinary differential equations in state-space form,

$$x'(t)=f(t,x(t),u(t));\ \ x(t_0)=x_0. \tag{11}$$

Here, $t\in\mathbb{R}$ stands for the independent variable, usually called time. In the case where f does not depend explicitly on t, the system is said to be autonomous. The vector $u(t)\in U$ represents the control variables at the time instant t. The vector $x(t)\in\mathbb{R}^{n_x}$ represents the state (or phase) variables which characterize the behavior of the system at any time instant t. A solution $x(t,x_0,u(.))$ of (11) is called a solution of the system, corresponding to the control $u(.)$, for the initial condition $x(t_0)=x_0$.

A differential motor robot contains two main wheels, each joining a motor separately. The third wheel in a differential motor robot is a spheroid wheel that is able to rotate in any direction; its only role is to keep the robot stable.

Dubins car is a simple model of a differential robot, moving forwards with the constant speed 1 and the minimum radius of rotating to left and right or equivalently its maximum curvature is equal to 1, cf. [10]. Consequently, the minimum rotating radius to left and right for Dubins airplane in the space is equal to the quantity of 1.

The Dubins airplane is a 3-dimentional extension of Dubins car. In this case the parameter z of altitude is also added to the system. This plane is a simple cinematic real airplane. It always flies forward and the system has independent bounded control over the altitude velocity as well as the turning rate in the plane. The Dubins airplane is a four-dimensional system with its configuration variable denoted by

$$q=(x,y,z,\theta)\in M=\mathbb{R}^3\times S^1,$$

In which x, y and z are the coordinates of the airplane's position in the three-dimensional Euclidian space $\mathbb{R}^3$. Let $\theta\in[0,2\pi)$ is the angle between x-axis of the frame and the airplane local line of site axis in $xy-$plane. Let $dz/dt=u_z$, be the speed vector in the direction of z-axis, $d\theta/dt=u_\theta$, be the plane's rotation speed and $dr/dt=u_r$, the projection of speed vector on the (x,y,θ)-plane.

The system has independent bounded control on u_θ, u_z and u_r. In other words, the system is

$$q'=f_0(q)+u_z f_1(q)+u_\theta f_2(q)+u_r f_3(q), \tag{12}$$

where f_0, f_1, f_2 and f_3 are vector fields in the tangent bundle TM of the configuration space. In this case, f_0, f_1, f_2 and f_3 are,

$$f_0=\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\\ 0\\ 0\end{pmatrix}, f_1=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, f_2=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\ and\ f_3=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}. \tag{13}$$

In above equations, (x,y,z,θ) are state variables and u_z, u_θ and u_r the system's control parameters. We assume the minimum turning radius and the maximum altitude velocity of the airplane are 1. For simplicity we assume the projection speed of the airplane on the (x,y,θ)-plane is constant and equal to 1, that is, $u_r=0$. We assume also that $|u_z|,|u_\theta|\leq 1$ and $u_r=0$. Thus, the control region is the square $U=[-1,1]^2$ and $u=(0,u_\theta,u_z)\in U$.

3.1. *Introducing a new metric*

Suppose (M,Φ), is a Finsler space, we put,

$$\Phi(x,v):=\|v\|=\sqrt{<v.v>}=\sqrt{v.v}, \tag{14}$$

Suppose an object (for example an airplane), travels throughout the vector v with unit speed. It is obvious that the time of traveling the length of the vector v, is equal to the $\|v\|$. Assume that an external factor ω produces an effect on M such that $\|w\|<1$. For example, here we assume that w is summation of control parameters of the airplane where length of w is smaller than 1. When the vector field w is present, the object (airplane) will travel through the resulting vector $v=u+w$ instead of u, in a unit of time. Before considering effect of the vector field w , the unit tangent sphere in any $T_x M$, includes all the tangent vectors u , so that $\|u\| = 1$. After considering effect of w , the sphere S_Φ, with mapping $u \to u + w$, changes to a convex domain.

This observation helps us to introduce a new norm or metric related to the new situation. This metric measures traveling time in presence of the control restrictions w .
Let F be a function that measures time of motion, then $\forall v \in T_x M$ we have

$$F(v) = 1.$$

Let $\Phi(x,u)$ be the length of the vector u at the point x defined in the section 2.2, then if length of u is equal one we have

$$\Phi(x,u) = \|u\| = \sqrt{u.u} = 1. \tag{15}$$

Therefore, (5), (6) and (7) imply

$$\Phi(x, \frac{v}{F} - w) = 1 \Rightarrow \left\| \frac{v}{F} - w \right\|^2 = 1. \tag{16}$$

So we have,

$$\left\| \frac{\upsilon}{F} \right\|^2 - 2\frac{\upsilon}{F}.\omega + \|\omega\|^2 = 1, \tag{17}$$

By multiplying F^2 , we obtain

$$F^2(1 - \|w\|^2) + 2v.wF - \|v\|^2 = 0. \tag{18}$$

Solving above second order equation for F we have

$$F = \frac{-<v.w>}{1-\|w\|^2} + \frac{\sqrt{<v.w>^2 + (1-\|w\|^2)\|v\|^2}}{1-\|w\|^2}. \tag{19}$$

It can be shown that F satisfies conditions of a Finsler metric called here, Randers metric, cf. [8], page 396.

Now, we are in a position to study the airplane's motion from Finsler geometry point of view. Dubins airplane always moves forward. When the airplane's control parameters produce an effect on the system, they cause the plane to take off, get out of the straight line, climbs in the direction of z -axis and travel through the paths with radius of $R \ge 1$, in any point, when moving. In other words; the airplane's control parameters produce an effect on the system as an external factor.
As it is remarked before, the Dubins airplane is a four-dimensional system with its configuration variable denoted by

$$q = (x, y, z, \theta) \in M = \mathbb{R}^3 \times S^1,$$

In which x , y and z are the coordinates of the airplane in the three-dimensional Euclidian space, and $\theta \in [0, 2\pi)$ is the angle between x -axis of the frame and the airplane local longitudinal axis in $xy-$plane. As stated, the control parameters produce an effect on the system as an external factor. The control parameters of the airplane system are similar to that of Dubins car together with an additional parameter in direction of z -axis. Without loss of generality we may assume, throughout this paper, which the initial configuration of the system at the point of beginning of the process of control of the airplane is $(0,0,0,0)$. We also denote the goal configuration by $(x_g, y_g, z_g, \theta_g)$. We distinguish three cases: low, medium, and high goal altitudes of the airplane. In fact, the final altitude plays a major role. In order to precisely define each case we give the following definitions.

Definition 3.1.1. Let Δ be the Dubins distance of (x_g, y_g, θ_g) from $(0,0,0)$. More precisely, let Δ denote the duration, or equivalently the length of the shortest Dubins curve from $(0,0,0)$ to (x_g, y_g, θ_g) . We call the final altitude low if $|z_g| \le \Delta$, medium if $\Delta \le |z_g| \le \Delta + 2\pi$, and high if $|z_g| \ge \Delta + 2\pi$,cf. [11].

For finding the time optimal paths, the cost functional J to be minimized is time, that is $J(u) := \int_0^T dt$. For every pair of initial and goal configuration, we seek an admissible control, that is a measurable function $u : [0,T] \to U$, which minimizes J while transferring the initial configuration to the goal configuration. Thus, it is viable to use the Pontryagin Maximum Principle (PMP) for this problem.
Let the Hamiltonian $H : \mathbb{R}^4 \times M \times U \to \mathbb{R}$ be

$$H(\lambda, q, u) = <\lambda, q'> \tag{20}$$

In which q' is given in (12). According to the PMP ,cf. [13], for every optimal trajectory $q(t)$ defined on $[0,T]$ and associated with control $u(t)$, there exists a constant $\lambda_0 \ge 0$ and an absolutely continuous vector-valued adjoint function $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$, which is nonzero if $\lambda_0 = 0$, with the following properties along the optimal trajectory:

$$\lambda' = -\frac{\partial H}{\partial q}, \tag{21}$$

$$H(\lambda(t), q(t), u(t)) = \max_{z \in U} H(\lambda(t), q(t), z), \tag{22}$$

$$H(\lambda(t), q(t), u(t)) \equiv \lambda_0. \tag{23}$$

Definition 3.1.2. An extremal is a trajectory $q(t)$ that satisfies the conditions of the PMP ,cf. [13].

In this section, let $q(t)$ be an extremal associated with the adjoint $\lambda(t)$ and the control $u(t)$. Equation (21) can be solved for λ to obtain

$$\lambda(t) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_1 y - c_2 x + c_4 \end{pmatrix} \tag{24}$$

In which c_1, c_2, c_3, and c_4 are constants. Along an extremal, (22) yields the extremal control law

$$u_z = \operatorname{sgn}(c_3) \;\; if \;\; c_3 \neq 0 \tag{25}$$

$$u_z \in [-1,1] \;\; if \;\; c_3 = 0 \tag{26}$$

$$u_\theta = \operatorname{sgn}(c_1 y - c_2 x + c_4) \;\; if \;\; c_1 y - c_2 x + c_4 \neq 0 \tag{27}$$

$$u_\theta \in [-1,1] \;\; if \;\; c_1 y - c_2 x + c_4 = 0 \tag{28}$$

If $c_3 = 0$, then (25) implies that u_z can have any value within [-1,1]. As it was stated, the final altitude plays a major role. So we distinguish three cases. We need following lemmas in the proof,

Lemma 3.1.1. For a low goal altitude, a time-optimal trajectory for the system (12) consists of the shortest Dubins curve with altitude velocity $u_z = \frac{z_g}{\Delta}$, cf. [11].

Lemma 3.1.2. For a high goal altitude, a time-optimal trajectory for the system (12) is composed of two pieces. Along both pieces $u_z = \operatorname{sgn}(z_g)$. The projection of the first piece on to the (x, y, θ)-space is the shortest Dubins curve for (x_g, y_g, θ_g). The second piece is a helix. The control is $u_\theta = \frac{2\pi}{|z_g| - \Delta}$ along the second piece, cf. [11].

Theorem 3.1.1. Let the Dubins airplane starts from the point $(x_0, y_0, z_0, \theta_0)$ in a non-obstacle space, in order to reach the point $(x_g, y_g, z_g, \theta_g)$. We assume that the system has independent bounded control over the altitude velocity as well as the turning rate in the plane. Then geometry of its movement is a special Finsler geometry called Randers geometry and time optimal paths are geodesics of a Randers metric.

Proof. We consider three steps for the proof related to the low, medium and high goal altitudes of the airplane and calculate the metric of the airplane's motion in each case.

Step 1. Low Goal Altitude

As it is mentioned on [11], the shortest Dubins curve with an unsaturated altitude velocity is a time-optimal strategy for low goal altitudes. This case corresponds to $c_3 = 0$ in the PMP analysis in (24). Note that the duration of such trajectory is Δ. It is obvious that there exists no trajectory transferring the system faster from the initial configuration to the goal configuration.

Using lemma (3.1.1), the extremal control low for low goal altitude is

$$u_\theta^* = \begin{cases} 1 & c_1 y - c_2 x + c_4 < 0 \\ -1 & c_1 y - c_2 x + c_4 < 0 \\ \in [-1,1] & c_1 y - c_2 x + c_4 < 0 \end{cases} \tag{29}$$

$$u_z^* = \frac{z_g}{\Delta} \tag{30}$$

As it was mentioned before, the control parameters produce an effect on the system as an external factor. The control parameters of the airplane system are similar to that of Dubins car together with an additional parameter in direction of z-axis. The control parameter u_z permits the airplane to increase or decrease the altitude and u_θ let it turns left and right. So we can consider the control parameters as a vector field w of class $\mathbb{C}^r$, acting on the airplane.

Hence, in a cylindrical coordinate, the external factor can be considered as

$$w = (u_r, u_\theta, u_z),$$

Where for simplicity we have assumed that $u_r = 0$.

By (29) and (30), the extremal tangent vector (tangent vector on the extremal trajectory for low goal altitude) is

$$v = (0, u_\theta^*, \frac{z_g}{\Delta}),$$

Therefore, presence of vector w causes the metric of the plane's movement to changes from Euclidean to Finslerian, as follows.

By replacing $w = (0, u_\theta, u_z)$ and $v = (0, u_\theta^*, \frac{z_g}{\Delta})$ in (19), we have

$$< v, w >= u_\theta^* . u_\theta + \frac{z_g}{\Delta} u_z$$

Also,

$$|v|^2 = (u_\theta^*)^2 + (\frac{z_g}{\Delta})^2 \;, \; |w|^2 = (u_\theta)^2 + (u_z)^2$$

Consequently, Eq. (19) becomes

$$\tilde{F} = \frac{-u_\theta^* . u_\theta - \frac{z_g}{\Delta} u_z}{1 - u_\theta^2 - u_z^2} + \frac{\sqrt{u_\theta^* . u_\theta + \frac{z_g}{\Delta} u_z + (1 - u_\theta^2 - u_z^2)((u_\theta^*)^2 + (\frac{z_g}{\Delta})^2)}}{1 - u_\theta^2 - u_z^2}.$$

The resulting metric is a special Finsler metric known in the literature as Randers metric. So $\tilde{F}$, the metric of the airplane motion for low goal altitude, is the above Randers metric.

Step 2. High Goal Altitude

If the goal altitude is high, the system has enough time to follow a helix once it reaches the goal point in the plane and goal orientation. Hence, the shortest Dubins curve followed by a helix all with saturated altitude velocity is a time-optimal strategy in this case. This case corresponds to $c_3 \neq 0$ in the PMP analysis in (25). The duration of such trajectory is $|z_g|$. There exists no trajectory taking the system faster from the initial to the goal.

The system first traverses the shortest Dubins curve with saturated altitude velocity along such time-optimal trajectory. It then traverses a helix, that is a full circle in the plane with saturated altitude velocity.

So according to the lemma (5.1.2), in the first part of the extremal path, the extremal control low for high goal altitude is

$$u_\theta^* = \begin{cases} 1 & c_1 y - c_2 x + c_4 < 0 \\ -1 & c_1 y - c_2 x + c_4 < 0 \\ \in [-1,1] & c_1 y - c_2 x + c_4 < 0 \end{cases} \tag{31}$$

$$u_z^* = \operatorname{sgn}(z_g) \tag{32}$$

Hence, in a cylindrical coordinate, the external factor can be considered as

$$w = (0, u_\theta, u_z),$$

By (31) and (32), the extremal tangent vector (tangent vector on the first part of the extremal trajectory for high goal altitude) is

$$v = (0, u_\theta^*, \operatorname{sgn}(z_g)),$$

By replacing $w = (0, u_\theta, u_z)$ and $v = (0, u_\theta^*, \operatorname{sgn}(z_g))$ in (19), we have

$$< v, w >= u_\theta^* . u_\theta + \operatorname{sgn}(z_g) u_z$$

Also,

$$|v|^2 = (u_\theta^*)^2 + (\operatorname{sgn}(z_g))^2 \ , \ |w|^2 = (u_\theta)^2 + (u_z)^2$$

Consequently, Eq. (19) becomes

$$\tilde{F}_1 = \frac{-u_\theta^* . u_\theta - \operatorname{sgn}(z_g) . u_z}{1 - u_\theta^2 - u_z^2} +,$$

$$\frac{\sqrt{u_\theta^* . u_\theta + \operatorname{sgn}(z_g) . u_z + (1 - u_\theta^2 - u_z^2)((u_\theta^*)^2 + (\operatorname{sgn}(z_g))^2)}}{1 - u_\theta^2 - u_z^2}.$$

Similarly, for the second part of the extremal path, the extremal control low for high goal altitude is

$$u_\theta^* = \frac{2\pi}{|z_g| - \Delta} \tag{33}$$

$$u_z^* = \operatorname{sgn}(z_g) \tag{34}$$

So, Eq. (19) becomes

$$\tilde{F}_2 = \frac{\dfrac{2\pi u_\theta}{|z_g| - \Delta} - \operatorname{sgn}(z_g) u_z}{1 - u_\theta^2 - u_z^2} +,$$

$$\frac{\sqrt{\dfrac{2\pi u_\theta}{|z_g| - \Delta} + \operatorname{sgn}(z_g) u_z + (1 - u_\theta^2 - u_z^2)\left(\dfrac{4\pi^2}{(|z_g| - \Delta)^2} + (\operatorname{sgn}(z_g))^2\right)}}{1 - u_\theta^2 - u_z^2}.$$

Consequently, for a high goal altitude, the time-optimal trajectory for the airplane is composed of two pieces. In the first part of path the metric of the airplane motion is $\tilde{F}_1$ and in the second part, $\tilde{F}_2$ is the metric of the movement. Obviously, both $\tilde{F}_1$ and $\tilde{F}_2$ are Finsler metrics.

Step 3. Medium Goal Altitude

If there is a path for the Dubins car from the initial configuration to the goal configuration in time $|z_g|$, then the time-optimal trajectory for the system corresponds to $c_3 \neq 0$ in the PMP analysis in (25). In this case the altitude velocity is saturated. If there is no path for the Dubins car from the initial configuration to the goal configuration in time $|z_g|$, then the time-optimal trajectory for the system must correspond to $c_3 = 0$ in the PMP analysis in (26). The altitude velocity is not saturated in this case. Thus, the projection of the time-optimal trajectory on to the (x, y, θ)-space is a Dubins time-extremal. Dubins time-extremals are composed of turn with minimum radius and straight line segments, cf. [11].

Hence, for medium goal altitude, if there is a path for the Dubins car from the initial configuration to the goal configuration in time $|z_g|$, a time-optimal trajectory for the system is composed of two pieces. The metric of the airplane's motion in the first part of the path corresponds to $\tilde{F}_1$ and in the second part corresponds to $\tilde{F}_2$.

If there is no path for the Dubins car from the initial configuration to the goal configuration in time $|z_g|$, the metric of the airplane's motion corresponds to $\tilde{F}$.

□

4. CONCLUSIONS

Using system control parameters of Dubins airplane as an external factor which affects the system, geometry of airplane's movement was described and a metric was also found.

This allowed us to study control system of Dubins airplane as a geometric problem and to use features of this geometry properly; i.e., geodesics of this metric are time optimal paths for Dubins airplane.

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