Amirkabir University of Technology
(Tehran Polytechnic)
Vol. 46, No. 2, Fall 2014, pp. 23- 33

Amirkabir International Journal of Science& Research
(Electrical & Electronics Engineering)
(AIJ-EEE)

Derivation of Green's Function for the Interior Region of a Closed Cylinder

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ABSTRACT

The importance of constructing the appropriate Green function to solve a wide range of problems inelectromagnetics and partial differential equations is well-recognized by those dealing with classical electrodynamics and related fields. Although the subject of obtaining the Green function for certain geometries has been extensively studied and addressed in numerous sources, in this paper a systematic method using the Method of Separation of Variables has been employed to scrutinize the Green function with Dirichlet boundary condition for the interior region of a closed cylinder. With further rigorous elaboration, we have demonstrated clearly the path through which the Green function can be accomplished. Additional verifications both in analytical and computer-simulating problems have also been performed to demonstrate the validity of our analysis.

KEYWORDS

Cylindrical Green's Function, Electrostatic Potential, Poisson's Equation.

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1- INTRODUCTION

As discussed extensively in numerous textbooks [1, 2, 3, 4] and papers [5, 6, 7], once we have obtained the correct Green function for a particular geometry, the problem of finding the electro-static potential (and hence the electric field) for all problems with that particular geometry is reduced to the trivial problem of solving the integrals in the following equation with the specified boundary conditions [1]:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho\left(\vec{x'}\right) G\left(\vec{x},\vec{x'}\right) d^3x' + \frac{1}{4\pi}\oiint_S \left[G\left(\vec{x},\vec{x'}\right)\frac{\partial\Phi\left(\vec{x'}\right)}{\partial n'} - \Phi\left(\vec{x'}\right)\frac{\partial G\left(\vec{x},\vec{x'}\right)}{\partial n'}\right] da' \quad (1)$$

In general, a Green function for Dirichlet boundary condition (DBC) satisfies the following equation [2]:

$$\nabla_x{}^2 G\left(\vec{x},\vec{x'}\right) = -4\pi\delta\left(\vec{x}-\vec{x'}\right) \quad (2)$$

where $G_D\left(\vec{x},\vec{x'}\right) = 0$ for $\vec{x}$ on the boundary surface of the cylinder which we denote by S.

Our objective here is to construct the proper Green function for the interior region of a closed cylindrical surface with DBC. For our problem, we consider a cylinder with height l and radius a as shown in Fig.1.

It is necessary to mention that there are two possible approaches to achieve the appropriate Green function for the geometry introduced. Elaborating upon both approaches, we will expand on how each of these approaches leads to a particular solution that is apparently different in form from the other one, nevertheless mathematically equivalent to each other.

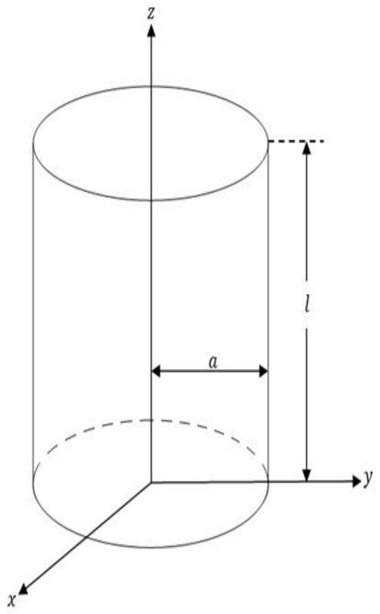

Fig. 1. Conducting cylinder of height l and radius a.

It must also be said that during all the derivations in this paper, by primed quantities such as $\vec{x'}$, we refer to the source points and by unprimed quantities such as $\vec{x}$ we refer to the observation points. Conventional notation of cylindrical coordinate system (ρ, ϕ, z) has been used throughout the text.

It is worth noting that the concept of constructing appropriate Green's functions for different geometries is becoming increasingly important in today's analysis of engineering problems such as antenna design, Composite Right/Left-Handed transmission lines for more exact solutions and simulations. [8], [9].

2- CONSTRUCTING THE GREEN FUNCTION, FIRST APPROACH

Cylindrical coordinate system is one of the eleven different coordinate systems in which the Laplacian operator is known to be separable [2]. Thus we can separate the corresponding Green function as follows:

$$G\left(\vec{x},\vec{x'}\right) = G(\rho,\rho',\phi,\phi',z,z') = R(\rho,\rho')Q(\phi,\phi')Z(z,z') \quad (3)$$

The Laplacian operator in the cylindrical coordinate system is defined as:

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2} + \frac{\partial^2 G}{\partial z^2} \quad (4)$$

Replacing (3) and (4) in (2) we get:

$$\frac{1}{\rho}\frac{R'}{R} + \frac{R''}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} + \frac{Z''}{Z} = 0, \quad \rho \neq 0 \quad (5)$$

At this stage, two mathematically valid options of allocating separation constants are available. Here, we evaluate both options and examine the results.

First, it is conventional to assign the positive number $+k^2$ to the term $\frac{Z''}{Z}$, where k is an arbitrary number leading to:

$$Z'' - k^2 Z = 0 \quad (6)$$

$$\frac{1}{\rho}\frac{R'}{R} + \frac{R''}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} = -k^2 \quad (7)$$

After appropriate rearrangements in the equation (7) and assigning separation constant $-v^2$ to the term $\frac{Q''}{Q}$ where v is an arbitrary number, combined with (6), we obtain three ordinary differential equations (ODEs):

$$Q'' + v^2 Q = 0 \quad (8)$$

$$\rho^2 R'' + \rho R'(\rho) + (k^2\rho^2 - v^2)R = 0 \quad (9)$$

At this point, we particularize solutions of each of the aforementioned ODEs.

Rewriting equation (6) in a more comprehensive form, we have:

$$\frac{d^2 Z(z,z')}{dz^2} - k^2 Z(z,z') = 0 \tag{10}$$

Dictating the general solutions of the above-mentioned ODE and also basic concepts of Green function method we arrive at:

$$Z_k(z,z') = \begin{cases} A_k(z')\sinh(kz) + B_k(z')\cosh(kz) & 0 < z < z' < l \\ C_k(z')\sinh(kz) + D_k(z')\cosh(kz) & 0 < z' < z < l \end{cases} \tag{11}$$

Enforcing the boundary condition $Z_k(z,z')\big|_{z=0} = 0$, leads to $B_K(z') = 0$.

Applying the second boundary condition, namely $Z_k(z,z')\big|_{z=l} = 0$, leads to:

$$C_k(z') = -\frac{\cosh(kl)}{\sinh(kl)} D_K(z') \tag{12}$$

Additionally, there are two more restrictions on the Green function which lead to establishing the other two coefficients. The first restriction is the continuity theorem which states that the Green function is continuous around the source point [10]. It is important to emphasize that the continuity theorem also holds for each of the components of a Green function separately in general. As a result:

$$Z_K(z,z')\big|_{z=z'^+} = Z_K(z,z')\big|_{z=z'^-} \tag{13}$$

which leads to:

$$A_k(z') = D_k(z')\left(\frac{\cosh(kz')}{\sinh(kz')} - \frac{\cosh(kl)}{\sinh(kl)}\right) = D_K(z')\frac{\sinh(k(l-z'))}{\sinh(kz')\sinh(kl)} \tag{14}$$

The second restriction is the symmetry of Green function between source points and observation points, both in its general form and also for each of its components similar to the continuity theorem [3]. Mathematically, for the Green Function with DBC, we have:

$$G_D(\vec{x},\vec{x'}) = G_D(\vec{x'},\vec{x}) \tag{15}$$

Generally we enforce this rule to the system of solutions, by equating one solution in the system with the other one with its primed and unprimed parameters exchanged, that is:

$$D_K(z)\frac{\sinh(k(l-z'))}{\sinh(kl)} = D_K(z')\frac{\sinh(k(l-z'))}{\sinh(kz')\sinh(kl)}\sinh(kz) \tag{16}$$

in which we have equated the first solution with the second one with primed and unprimed parameters exchanged in the second one. Hence:

$$D_K(z)\sinh(kz') = D_K(z')\sinh(kz) \Rightarrow D(x) = \sinh(kx) \tag{17}$$

Having determined all the coefficients, we arrive at the following expression for $Z_k(z,z')$:

$$Z_k(z,z') = \begin{cases} \dfrac{\sinh(kz)\sinh(k(l-z'))}{\sinh(kl)} & 0 < z < z' < l \\ \dfrac{\sinh(k(l-z))\sinh(kz')}{\sinh(kl)} & 0 < z' < z < l \end{cases} \tag{18}$$

The next step is to derive the solution for the ODE involving function $Q_v(\phi,\phi')$. Rewriting the ODE in its more complete form we have:

$$\frac{d^2 Q(\phi,\phi')}{d\phi^2} + v^2 Q(\phi,\phi') = 0 \tag{19}$$

Again the general solution can be written as:

$$Q_v(\phi,\phi') = \begin{cases} A_v(\phi')\sin(v\phi) + B_v(\phi')\cos(v\phi) & 0 < \phi < \phi' < 2\pi \\ C_v(\phi')\sin(v\phi) + D_v(\phi')\cos(v\phi) & 0 < \phi' < \phi < 2\pi \end{cases} \tag{20}$$

Applying the continuity theorem and symmetry principle and considering the mathematical independence of sine and cosine functions, all the coefficients are obtained:

$$Q_v(\phi,\phi')\big|_{\phi=\phi'^+} = Q_v(\phi,\phi')\big|_{\phi=\phi'^-} \Rightarrow \begin{cases} A_v(\phi') = C_v(\phi') \\ B_v(\phi') = D_v(\phi') \end{cases} \tag{21}$$

$$A_v(\phi)\sin(v\phi') + B_v(\phi)\cos(v\phi') = A_v(\phi')\sin(v\phi) + B_v(\phi')\cos(v\phi) \Rightarrow \begin{cases} A_v(x) = \sin(vx) \\ B_v(x) = \cos(vx) \end{cases} \tag{22}$$

Combining the results achieved in (21) and (22), we can write $Q_v(\phi,\phi')$ in closed form:

$$Q_v(\phi,\phi') = \begin{cases} \sin(v\phi')\sin(v\phi) + \cos(v\phi')\cos(v\phi) & 0 < \phi < \phi' < 2\pi \\ \sin(v\phi')\sin(v\phi) + \cos(v\phi')\cos(v\phi) & 0 < \phi' < \phi < 2\pi \end{cases}$$

$$\Rightarrow Q_v(\phi,\phi') = \cos v(\phi - \phi') \tag{23}$$

Regarding the fact that the cylinder is complete and values for $Q_v(\phi,\phi')$ must be periodic with period 2π, we can write:

$$Q_v(\phi,\phi') = Q_v(\phi+2\pi,\phi') \Rightarrow \cos v(\phi-\phi') = \cos(v(\phi-\phi')+2\pi v) \Rightarrow v = m \quad (m\in\mathbb{Z}) \tag{24}$$

The last stage of solving ODEs is to obtain the solution for the following ODE:

$$\rho^2\frac{d^2R(\rho,\rho')}{d\rho^2}+\rho\frac{dR(\rho,\rho')}{d\rho}+(k^2\rho^2-v^2)R(\rho,\rho')=0 \tag{25}$$

This is the Bessel equation whose corresponding solution is in the form of:

$$R_{v,k}(\rho,\rho') = \begin{cases} A_{v,k}(\rho')J_v(k\rho)+B_{v,k}(\rho')Y_v(k\rho) & 0<\rho<\rho'<a \\ C_{v,k}(\rho')J_v(k\rho)+D_{v,k}(\rho')Y_v(k\rho) & 0<\rho'<\rho<a \end{cases} \tag{26}$$

Where $J_v(k\rho)$ and $Y_v(k\rho)$ correspond to the Bessel functions of first and second kind respectively.

The Bessel function of the second kind has a singularity at the origin which is included in the domain of our problem [11]; hence, its corresponding coefficient $B_{v,k}(\rho')$ must be zero. Enforcing the continuity theorem to the solutions of system of (26) requires that the coefficient $D_{v,k}(\rho')$ is also to be zero due to the linear independence of Bessel functions of the first and second kind.

The function $R_{m,k}(\rho,\rho')$ is in the following form:

$$R_{m,k}(\rho,\rho') = \begin{cases} A_{m,k}(\rho')J_m(k\rho) & 0<\rho<\rho'<a \\ C_{m,k}(\rho')J_m(k\rho) & 0<\rho'<\rho<a \end{cases} \tag{27}$$

Enforcing the continuity theorem, we arrive at:

$$R_{m,k}(\rho,\rho')\big|_{\rho=\rho'^+} = R_{m,k}(\rho,\rho')\big|_{\rho=\rho'^-} \Rightarrow A_{m,k}(\rho') = C_{m,k}(\rho') \tag{28}$$

Finally, by employing the symmetry principle we can determine the coefficients $A_{m,k}(\rho')$ and $C_{m,k}(\rho')$ as follows:

$$A_{m,k}(\rho') = C_{m,k}(\rho') = J_m(k\rho') \tag{29}$$

Enforcing our last boundary condition leads to:

$$R_{m,k}(\rho,\rho')\big|_{\rho=a} = J_m(k\rho')J_m(ka) = 0 \Rightarrow J_m(ka)=0 \Rightarrow ka = x_{mn}$$

$$\Rightarrow k = \frac{x_{mn}}{a} \quad (n\in N) \tag{30}$$

in which x_{mn} denotes the n^th root of the Bessel function of the first kind and m^th order. This particular value of k must be replaced in all the previous equations.

Eventually, we can claim that the Green function we are seeking can be written as the linear combinations of all separate functions previously found in the form:

$$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} A_{mn}R_{m,n}(\rho,\rho')Z_{m,n}(z,z')\cos(m(\phi-\phi')) \tag{31}$$

with coefficients A_{mn} to be determined.

Returning our attention back to equation (2), the Dirac delta function in the right-hand- side of the equation can be separated in cylindrical coordinate system as follows:

$$\delta(\vec{x}-\vec{x}') = \frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho} \tag{32}$$

Replacing (32) in (2) and integrating both sides over the infinitesimal interval of (z'^-, z'^+), we can write:

$$\begin{aligned}&\int_{z=z'^-}^{z=z'^+}\nabla^2 G(\vec{x},\vec{x}')\,dz = -\int\frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho}dz\\ &\Rightarrow \int_{z=z'^-}^{z=z'^+}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2}+\frac{\partial^2 G}{\partial z^2}\right)dz\\ &= \underbrace{\int_{z=z'^-}^{z=z'^+}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2}\right)dz}_{zero}\\ &+\int_{z=z'^-}^{z=z'^+}\frac{\partial^2 G}{\partial z^2}dz \Rightarrow \frac{\partial G}{\partial z}\Big|_{z=z'^+}-\frac{\partial G}{\partial z}\Big|_{z=z'^-} = -\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho}\end{aligned} \tag{33}$$

Here we investigate further to determine the value of the left-hand-side of the equation (33). Taking the derivative of the Green function of (31) we have:

$$\frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\left[\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}A_{mn}R_{m,n}(\rho,\rho')Z_{m,n}(z,z')\cos(m(\phi-\phi'))\right]$$

$$= \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}A_{mn}R_{m,n}(\rho,\rho')\cos(m(\phi-\phi'))\frac{\partial(Z_{m,n}(z,z'))}{\partial z} \tag{34}$$

Hence:

$$\frac{\partial(Z_{m,n}(z,z'))}{\partial z}\Big|_{z=z'^+}-\frac{\partial(Z_{m,n}(z,z'))}{\partial z}\Big|_{z=z'^-}$$
$$=-\frac{x_{mn}}{a}\frac{\cosh\left(\frac{x_{mn}}{a}(l-z')\right)\sinh\left(\frac{x_{mn}}{a}z'\right)}{\sinh\left(\frac{x_{mn}}{a}l\right)}$$
$$-\frac{x_{mn}}{a}\frac{\cosh\left(\left(\frac{x_{mn}}{a}\right)z'\right)\sinh\left(\frac{x_{mn}}{a}(l-z')\right)}{\sinh\left(\frac{x_{mn}}{a}l\right)}$$
$$=-\frac{x_{mn}}{a}\frac{1}{\sinh\left(\frac{x_{mn}}{a}l\right)}\left[\sinh\left(\frac{x_{mn}}{a}z'\right)\cosh\left(\frac{x_{mn}}{a}(l-z')\right)\right.$$
$$\left.+\cosh\left(\left(\frac{x_{mn}}{a}\right)z'\right)\sinh\left(\frac{x_{mn}}{a}(l-z')\right)\right]$$
$$=-\frac{x_{mn}}{a}\frac{1}{\sinh\left(\frac{x_{mn}}{a}l\right)}\left[\sinh\left(\frac{x_{mn}}{a}z'+\frac{x_{mn}}{a}(l-z')\right)\right]$$
$$=-\frac{x_{mn}}{a}\frac{1}{\sinh\left(\frac{x_{mn}}{a}l\right)}\sinh\left(\frac{x_{mn}}{a}\right)=-\frac{x_{mn}}{a}$$
$$\Rightarrow$$
$$\frac{\partial G}{\partial z}\Big|_{z=z'^+}-\frac{\partial G}{\partial z}\Big|_{z=z'^-}=\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}A_{mn}R_{m,n}(\rho,\rho')\cos\left(m(\phi-\phi')\right)\left(-\frac{x_{mn}}{a}\right) \tag{35}$$

Accordingly, we can write:

$$\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}A_{mn}R_{m,n}(\rho,\rho')\cos\left(m(\phi-\phi')\right)\left(\frac{x_{mn}}{a}\right)=\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho} \tag{36}$$

Next, we multiply both sides of (36) by the term $\cos\left(p(\phi-\phi')\right)$ and integrate both sides over the interval (0, 2π):

$$\int_{\phi=0}^{\phi=2\pi}\sum_{n=1}^{+\infty}A_{mn}R_{m,n}(\rho,\rho')\cos\left(p(\phi-\phi')\right)\cos\left(m(\phi-\phi')\right)\left(\frac{x_{mn}}{a}\right)d\phi$$
$$=\int_{\phi=0}^{\phi=2\pi}\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho}\cos\left(p(\phi-\phi')\right)d\phi$$
$$=\frac{\delta(\rho-\rho')}{\rho} \tag{37}$$

Leading to:

$$\sum_{n=1}^{\infty}A_{pn}J_p\left(\frac{x_{pn}}{a}\rho\right)J_p\left(\frac{x_{pn}}{a}\rho'\right)(\pi)\left(\frac{x_{pn}}{a}\right)=\frac{\delta(\rho-\rho')}{\rho'} \tag{38}$$

And finally we multiply both sides of (38) by $\rho J_p\left(\frac{x_{pq}}{a}\rho\right)$ and integrate over the interval of (0, a):

$$\int_{\rho=0}^{\rho=a}\sum_{n=1}^{\infty}A_{pn}J_p\left(\frac{x_{pn}}{a}\rho\right)J_p\left(\frac{x_{pn}}{a}\rho'\right)\rho J_p\left(\frac{x_{pq}}{a}\rho\right)(\pi)\left(\frac{x_{pn}}{a}\right)d\rho$$
$$=\int_{\rho=0}^{\rho=a}\frac{\delta(\rho-\rho')}{\rho}\rho J_p\left(\frac{x_{pq}}{a}\rho\right)d\rho$$
$$\Rightarrow\sum_{n=1}^{\infty}A_{pn}J_p\left(\frac{x_{pn}}{a}\rho'\right)(\pi)\left(\frac{x_{pn}}{a}\right)\int_{\rho=0}^{\rho=a}\rho J_p\left(\frac{x_{pn}}{a}\rho\right)J_p\left(\frac{x_{pq}}{a}\rho\right)d\rho$$
$$=\sum_{n=1}^{\infty}A_{pn}J_p\left(\frac{x_{pn}}{a}\rho'\right)(\pi)\left(\frac{x_{pn}}{a}\right)\frac{a^2}{2}\left[J_{p+1}(x_{pn})\right]^2\delta_{n,q}$$
$$=A_{pq}J_p\left(\frac{x_{pq}}{a}\rho'\right)(\pi)\left(\frac{x_{pq}}{a}\right)\frac{a^2}{2}\left[J_{p+1}(x_{pq})\right]^2$$
$$=\frac{\pi a}{2}A_{pq}x_{pq}J_p\left(\frac{x_{pq}}{a}\rho'\right)\left[J_{p+1}(x_{pq})\right]^2$$
$$=\int_{\rho=0}^{\rho=a}\delta(\rho-\rho')J_p\left(\frac{x_{pq}}{a}\rho\right)d\rho=J_p\left(\frac{x_{pq}}{a}\rho'\right) \tag{39}$$

In deriving the previous equation, we have employed the identity [11]:

$$\int_{\rho=0}^{\rho=1}\rho J_p(x_{pn}\rho)J_p(x_{pq}\rho)d\rho=\frac{1}{2}\left[J_{p+1}(x_{pn})\right]^2\delta_{n,q} \tag{40}$$

with $\delta_{n,q}$ being the Kronecker delta function. Accordingly, the coefficient A_{pq} is determined as follows:

$$A_{pq}=\begin{cases}\dfrac{2}{\pi a x_{pq}\left[J_{p+1}(x_{pq})\right]^2} & \rho\neq 0\\ \dfrac{1}{\pi a x_{pq}\left[J_{p+1}(x_{pq})\right]^2} & \rho=0\end{cases} \tag{41}$$

Combining all these results, we can write the Green function:

$$G_D\left(\vec{x},\vec{x'}\right)=\sum_{n=1}^{+\infty}\frac{1}{\pi a x_{0n}[J_1(x_{0n})]^2}J_0\left(\frac{x_{0n}}{a}\rho'\right)J_0\left(\frac{x_{0n}}{a}\rho\right)\frac{\sinh(\frac{x_{0n}}{a}z_<)\sinh(\frac{x_{0n}}{a}(l-z_>))}{\sinh(\frac{x_{0n}}{a}l)}$$
$$+\sum_{\substack{m=-\infty\\m\neq0}}^{+\infty}\sum_{n=1}^{+\infty}\frac{2}{\pi a x_{mn}[J_{m+1}(x_{mn})]^2}J_m\left(\frac{x_{mn}}{a}\rho'\right)J_m\left(\frac{x_{mn}}{a}\rho\right)\frac{\sinh(\frac{x_{mn}}{a}z_<)\sinh(\frac{x_{mn}}{a}(l-z_>))}{\sinh(\frac{x_{mn}}{a}l)}\cos\left(m\left(\phi-\phi'\right)\right) \tag{42}$$

3- CONSTRUCTING THE GREEN FUNCTION, SECOND APPROACH

In this section, we use a similar approach as was employed in the previous section, with the exception of using $-k^2$ instead of k^2 as our separation constant.

Returning to the equation (5), we assigned $+k^2$ to the term $\frac{Z''}{Z'}$ while it was also possible to assign $-k^2$ to it. In this section, we want to demonstrate how the appearance of the Green function changes as a result of using this alternative approach. The two new equations resulting from this approach are:

$$Z'' + k^2 Z = 0 \tag{43}$$

$$\frac{1}{\rho}\frac{R'}{R} + \frac{R''}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} = +k^2 \tag{44}$$

with k being an arbitrary number. Again after a suitable rearrangement and assigning an arbitrary number like $-v^2$ to the term $\frac{Q''}{Q'}$ the other two ODEs are:

$$Q'' + v^2 Q = 0 \tag{45}$$

$$\rho^2 R'' + \rho R'(\rho) - (k^2\rho^2 + v^2)R = 0 \tag{46}$$

Equation (43) has a general solution of the form:

$$Z_k(z,z') = \begin{cases} A(z')\sin(kz) + B(z')\cos(kz) & 0 < z < z' < l \\ C(z')\sin(kz) + D(z')\cos(kz) & 0 < z' < z < l \end{cases} \tag{47}$$

with four unknown coefficients to be determined.

The coefficient $B_K(z')$ will be eliminated applying the boundary condition $Z_k(z,z')\big|_{z=0} = 0$. Applying the continuity theorem we can write:

$$Z_k(z,z')\big|_{z=z'+} = Z_k(z,z')\big|_{z=z'-} \Rightarrow A(z')\sin(kz') = C(z')\sin(kz') + D(z')\cos(kz') \tag{48}$$

Due to the linear independence of functions $\sin(kz')$ and $\cos(kz')$ we conclude that:

$$D_k(z') = 0 \tag{49}$$

And enforcing the symmetry principle, we can easily reason that:

$$A_k(z') = C_k(z') = \sin(kz') \tag{50}$$

Thus:

$$Z_k(z,z') = \begin{cases} \sin(kz')\sin(kz) & 0 < z < z' < l \\ \sin(kz')\sin(kz) & 0 < z' < z < l \end{cases} \tag{51}$$

The function $Q(\phi,\phi')$ is determined in the same manner as the first approach:

$$Q_v(\phi,\phi') = \cos v(\phi-\phi') \Rightarrow Q_v(\phi,\phi') = Q_v(\phi+2\pi,\phi')$$

$$\Rightarrow \cos v(\phi-\phi') = \cos(v(\phi-\phi') + 2\pi v) \tag{52}$$

$$v = m \quad (m \in \mathbb{Z})$$

Finally we are to find the proper solutions of the following ODE:

$$\rho^2 R'' + \rho R'(\rho) - (k^2\rho^2 + v^2)R = 0 \tag{46}$$

This is the Modified Bessel equation. The general solution to this equation is in the form of:

$$R_{v,k}(\rho,\rho') = \begin{cases} A_{v,k}(\rho')I_v(k\rho) + B_{v,k}(\rho')K_v(k\rho) & 0 < \rho < \rho' < a \\ C_{v,k}(\rho')I_v(k\rho) + D_{v,k}(\rho')K_v(k\rho) & 0 < \rho' < \rho < a \end{cases} \tag{53}$$

Enforcing the boundary condition $R_{m,k}(\rho,\rho')\big|_{\rho=0} = 0$, and considering the finiteness of Green function inside the cylinder [11], we can conclude that $B_{v,k}(\rho') = 0$. Enforcing leads to:

$$D_{v,k}(\rho') = -\frac{I_v(ka)}{K_v(ka)}C_{v,k}(\rho') \tag{54}$$

Writing the continuity theorem we have:

$$R_{m,k}(\rho,\rho')\big|_{\rho=\rho'+} = R_{m,k}(\rho,\rho')\big|_{\rho=\rho'-} \Rightarrow A_{v,k}(\rho')I_v(k\rho') = C_{v,k}(\rho')I_v(k\rho') + D_{v,k}(\rho')K_v(k\rho') \tag{55}$$

And the symmetry principle dictates:

$$A_{v,k}(\rho)I_v(k\rho') = C_{v,k}(\rho')I_v(k\rho) + D_{v,k}(\rho')K_v(k\rho') \tag{56}$$

Solving equations (54) to (56) simultaneously we will obtain all the coefficients:

$$C_{v,k}(\rho') = I_v(k\rho') \tag{57}$$

$$D_{v,k}(\rho') = -\frac{I_v(ka)}{K_v(ka)}I_v(k\rho') \tag{58}$$

$$A_{v,k}(\rho') = \frac{I_v(k\rho')K_v(ka) - K_v(k\rho')I_v(ka)}{K_v(ka)} \tag{59}$$

Inserting (57) to (59) into (53) we will have:

$$R_{v,k}(\rho,\rho') = \begin{cases} \frac{I_v(k\rho')K_v(ka) - K_v(k\rho')I_v(ka)}{K_v(ka)} I_v(k\rho) & 0<\rho<\rho'<a \\ I_v(k\rho')I_v(k\rho) - \frac{I_v(ka)}{K_v(ka)} I_v(k\rho')K_v(k\rho) & 0<\rho'<\rho<a \end{cases} \quad (60)$$

Here we aim to prove that k is equal to an integer times a constant. In the function $Z_k(z,z')$ found previously in (51), one boundary condition was left out and that was $Z_k(z,z')\big|_{z=l} = 0$. Enforcing this boundary condition we will have:

$$Z_k(z,z')\big|_{z=l} = 0 \implies \sin(kl) = 0$$
$$\implies kl = n\pi \quad (n \in \mathbb{Z}) \implies k = \frac{n\pi}{l} \quad (61)$$

Non-positive values of n can be ignored since they correspond to non-defined arguments for the modified Bessel functions.

Accordingly, the Green function can be written as follows:

$$G\left(\vec{x},\overrightarrow{x'}\right) = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} A_{mn} R_{m,n}(\rho,\rho') Z_m(z,z') \cos\left(m(\phi - \phi')\right) \quad (62)$$

with $R_{m,n}(\rho,\rho')$ and $Z_m(z,z')$ defined in (60) and (51) respectively, and the coefficients A_{mn} to be determined. We follow a similar procedure as was done in the previous section by inserting the separable form of the Dirac delta function in the cylindrical system in equation (2).

Next, we apply the Laplacian operator to the Green function of (62). Integrating both sides over the infinitesimal interval of (ρ'^{-},ρ'^{+}) leads to:

$$\underbrace{\int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \left(\frac{\partial^2 G}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \phi^2}\right) d\rho}_{zero} + \int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial G}{\partial \rho}\right) d\rho = -\int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho} d\rho \quad (63)$$

The non-zero integral in the left-hand-side of (63) can be further simplified to:

$$\int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial G}{\partial \rho}\right) d\rho = \int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \frac{1}{\rho}\left(\frac{\partial G}{\partial \rho} + \rho \frac{\partial^2 G}{\partial \rho^2}\right) d\rho = \underbrace{\int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \left(\frac{1}{\rho}\left(\frac{\partial G}{\partial \rho}\right)\right) d\rho}_{zero} + \int_{\rho=\rho'^{-}}^{\rho=\rho'^{+}} \frac{\partial^2 G}{\partial \rho^2} d\rho = \frac{\partial G}{\partial \rho}\Big|_{\rho=\rho'^{+}} - \frac{\partial G}{\partial \rho}\Big|_{\rho=\rho'^{-}} = -4\pi \frac{\delta(z-z')\delta(\phi-\phi')}{\rho'} \quad (64)$$

Replacing the Green function of (62) in left-hand side of (64) and after some manipulating, we arrive at:

$$\frac{\partial G}{\partial \rho}\Big|_{\rho=\rho'^{+}} = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} A_{mn} Z_{m,n}(z,z') \cos\left(m(\phi - \phi')\right) \frac{I_v(k\rho')K_v(ka) - K_v(k\rho')I_v(ka)}{K_v(ka)} \frac{\partial}{\partial \rho} I_v(k\rho)\Big|_{\rho=\rho'} \quad (65)$$

$$\frac{\partial G}{\partial \rho}\Big|_{\rho=\rho'^{-}} = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} A_{mn} Z_{m,n}(z,z') \cos\left(m(\phi - \phi')\right) I_v(k\rho')\left(\frac{\partial}{\partial \rho} I_v(k\rho)\Big|_{\rho=\rho'} - \left(\frac{I_v(ka)}{K_v(ka)}\right)\frac{\partial}{\partial \rho} K_v(k\rho)\Big|_{\rho=\rho'}\right) \quad (66)$$

Substituting the derived values of (65) and (66) in the left-hand-side of (64) it can be shown that:

$$\frac{\partial G}{\partial \rho}\Big|_{\rho=\rho'^{+}} - \frac{\partial G}{\partial \rho}\Big|_{\rho=\rho'^{-}} = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} A_{mn} Z_{m,n}(z,z') \cos\left(m(\phi - \phi')\right)\left(-\frac{1}{\rho'}\frac{I_v(ka)}{K_v(ka)}\right) \quad (67)$$

In deriving (67), we have employed the following identity between modified Bessel functions [11]:

$$I_v(k\rho)\frac{\partial}{\partial \rho} K_v(k\rho) - K_v(k\rho)\frac{\partial}{\partial \rho} I_v(k\rho) = -\frac{1}{\rho} \quad (68)$$

Inserting (67) in the left-hand- side of (64) and multiplying both sides by the term:

$$\sin\left(\frac{q\pi}{l}z\right)\cos\left(p\left(\phi-\phi'\right)\right) \tag{69}$$

we then integrate both sides twice. Once over the interval of (0, a) with respect to variable ρ and once over the interval of (0, 2π) with respect to variable ϕ, arriving at:

$$A_{pq}\left(-\frac{I_v(ka)}{K_v(ka)}\right)\sin\left(\frac{n\pi}{l}z'\right)\left(\frac{l}{2}\right)(\pi) = -4\pi\ \sin\left(\frac{n\pi}{l}z'\right)$$

$$\Rightarrow A_{pq} = \frac{8}{l}\frac{K_v(ka)}{I_v(ka)} \tag{70}$$

Finally the sought-after Green function can be written in its second closed-form as:

$$G\left(\vec{x},\vec{x'}\right) = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\frac{8}{l}\frac{K_m\left(\frac{n\pi}{l}a\right)}{I_m\left(\frac{n\pi}{l}a\right)}R_{m,n}(\rho,\rho')\sin\left(\frac{n\pi}{l}z'\right)\sin\left(\frac{n\pi}{l}z\right)\cos\left(m(\phi-\phi')\right)$$

$$\Rightarrow G\left(\vec{x},\vec{x'}\right) = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\frac{8}{l}\cos\left(m(\phi-\phi')\right)\sin\left(\frac{n\pi}{l}z'\right)\sin\left(\frac{n\pi}{l}z\right)\frac{I_m\left(\frac{n\pi}{l}\rho_>\right)}{I_m\left(\frac{n\pi}{l}a\right)}\left(I_m\left(\frac{n\pi}{l}\rho_<\right)K_m\left(\frac{n\pi}{l}a\right) - K_m\left(\frac{n\pi}{l}\rho_<\right)I_m\left(\frac{n\pi}{l}a\right)\right) \tag{71}$$

4- ANALYTICAL VERIFICATION

A famous problem usually raised in textbooks [1] when discussing boundary-value problems in cylindrical coordinates is shown in Fig .2.

The cylinder has a radius a and height l, with the top and bottom of the cylinder being at $z = 0$ and $= l$, respectively. The potential on the side and the bottom of the cylinder is zero, while the top has a potential $V(\rho', \phi')$.

Here we demonstrate the validity of our approach by evaluating the potential at any point inside the cylinder using our Green function and show that they are in complete agreement with the previously published results.

As previously mentioned, for such DBC problems, once the Green function is obtained, solution is no more than calculating the integrals below regarding the boundary conditions specified:

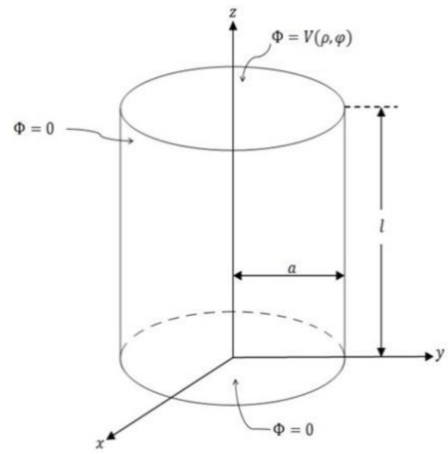

Fig. 2. Geometry of a Laplace problem for the interior region of a cylinder.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho\left(\vec{x'}\right)G\left(\vec{x},\vec{x'}\right)d^3x' + \frac{1}{4\pi}\oint_S\left[G\left(\vec{x},\vec{x'}\right)\frac{\partial\Phi\left(\vec{x'}\right)}{\partial n'} - \Phi\left(\vec{x'}\right)\frac{\partial G\left(\vec{x},\vec{x'}\right)}{\partial n'}\right]da' \tag{1}$$

Before we substitute the Green function of (71) in this equation, we make a few notes that will significantly reduce the amount of calculations.

1. There is no charge density in the problem. Therefore the first integral in (1) is eliminated.

2. In the second integral, which must be taken over the entire surface of the cylinder, because of the fact that the potential is zero everywhere on the surface except the top side, the second integral reduces to be taken only on the top face.

3. On the top side, the unit normal vector is:

$$\widehat{n'} = \hat{z} \tag{72}$$

Since the potential on the top side is not dependent on z', the term $\frac{\partial\Phi\left(\vec{x'}\right)}{\partial n'}$ is eliminated.

As a result, the whole right-hand-side of (1) reduces to evaluating the following integral:

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi\left(\vec{x'}\right)\frac{\partial G\left(\vec{x},\vec{x'}\right)}{\partial n'}da' \tag{73}$$

taken solely on the top side.

The important point here is that since the source point in this situation is the top side and all observation points are below that, they will all have a height smaller than the source points. Consequently:

$$z_< = z \tag{74}$$

$$z_> = z' \tag{75}$$

leading to:

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\int_{\rho'=0}^{\rho'=a}\int_{\phi'=0}^{\phi'=2\pi} V(\rho',\phi')\frac{\partial}{\partial z'}\left\{\sum_{n=1}^{+\infty}\frac{1}{\pi a x_{0n}[J_1(x_{0n})]^2}J_0\left(\frac{x_{0n}}{a}\rho'\right)J_0\left(\frac{x_{0n}}{a}\rho\right)\frac{\sinh\left(\frac{x_{0n}}{a}z_<\right)\sinh\left(\frac{x_{0n}}{a}(l-z_>)\right)}{\sinh\left(\frac{x_{0n}}{a}l\right)} + \sum_{\substack{m=-\infty\\m\neq 0}}^{+\infty}\sum_{n=1}^{+\infty}\frac{2}{\pi a x_{mn}[J_{m+1}(x_{mn})]^2}J_m\left(\frac{x_{mn}}{a}\rho'\right)J_m\left(\frac{x_{mn}}{a}\rho\right)\frac{\sinh\left(\frac{x_{mn}}{a}z_<\right)\sinh\left(\frac{x_{mn}}{a}(l-z_>)\right)}{\sinh\left(\frac{x_{mn}}{a}l\right)}\cos\left(m(\phi-\phi')\right)\right\}\Big|_{z'=l}\rho' d\rho' d\phi' \tag{76}$$

$$\Rightarrow \Phi(\vec{x}) = -\frac{1}{4\pi}\int_{\rho'=0}^{\rho'=a}\int_{\phi'=0}^{\phi'=2\pi} V(\rho',\phi')\left\{\sum_{n=1}^{+\infty}\frac{1}{\pi a x_{0n}[J_1(x_{0n})]^2}J_0\left(\frac{x_{0n}}{a}\rho'\right)J_0\left(\frac{x_{0n}}{a}\rho\right)\frac{\sinh\left(\frac{x_{0n}}{a}z_<\right)\left(\frac{x_{0n}}{a}\right)\cosh\left(\frac{x_{0n}}{a}(l-z_>)\right)}{\sinh\left(\frac{x_{0n}}{a}l\right)} + \sum_{\substack{m=-\infty\\m\neq 0}}^{+\infty}\sum_{n=1}^{+\infty}\frac{2}{\pi a x_{mn}[J_{m+1}(x_{mn})]^2}J_m\left(\frac{x_{mn}}{a}\rho'\right)J_m\left(\frac{x_{mn}}{a}\rho\right)\frac{\sinh\left(\frac{x_{mn}}{a}z_<\right)\left(\frac{x_{mn}}{a}\right)\cosh\left(\frac{x_{mn}}{a}(l-z_>)\right)}{\sinh\left(\frac{x_{mn}}{a}l\right)}\cos\left(m(\phi-\phi')\right)\right\}\Big|_{z'=l}\rho' d\rho' d\phi' \tag{77}$$

Now if we define three new parameters as follows:

$$A_{0n} = -\frac{1}{4\pi}\frac{1}{\pi a x_{0n}[J_1(x_{0n})]^2\sinh\left(\frac{x_{0n}}{a}l\right)}\int_{\rho'=0}^{\rho'=a}\int_{\phi'=0}^{\phi'=2\pi} V(\rho',\phi')J_0\left(\frac{x_{0n}}{a}\rho'\right)\rho' d\rho' d\phi' \tag{78}$$

$$A_{mn} = -\frac{1}{4\pi}\frac{2}{\pi a x_{mn}[J_{m+1}(x_{mn})]^2\sinh\left(\frac{x_{mn}}{a}l\right)}\int_{\rho'=0}^{\rho'=a}\int_{\phi'=0}^{\phi'=2\pi} V(\rho',\phi')J_m\left(\frac{x_{mn}}{a}\rho'\right)\sin(m\phi')\rho' d\rho' d\phi' \tag{79}$$

$$B_{mn} = -\frac{1}{4\pi}\frac{2}{\pi a x_{mn}[J_{m+1}(x_{mn})]^2\sinh\left(\frac{x_{mn}}{a}l\right)}\int_{\rho'=0}^{\rho'=a}\int_{\phi'=0}^{\phi'=2\pi} V(\rho',\phi')J_m\left(\frac{x_{mn}}{a}\rho'\right)\cos(m\phi')\rho' d\rho' d\phi' \tag{80}$$

then the potential of (73) can be written in a more compact form as follows:

$$\Phi(\vec{x}) = \sum_{n=1}^{+\infty} A_{0n}J_0\left(\frac{x_{0n}}{a}\rho\right)\sinh\left(\frac{x_{0n}}{a}l\right) + \sum_{\substack{m=-\infty\\m\neq 0}}^{+\infty}\sum_{n=1}^{+\infty} J_m\left(\frac{x_{mn}}{a}\rho\right)\sinh\left(\frac{x_{0n}}{a}l\right)\left(A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right) \tag{81}$$

which is in complete accordance with what is already derived in the literature [1].

5- NUMERICAL VERIFICATIONS

To further verify the validity of the Green function obtained in this paper we have simulated two separate problems in commercially available numerical software Computer Simulation Technology (CST EM STUDIO 2015) and compared the results with what we achieved through the use of the Green function constructed in this paper.

The first problem is the problem we employed in the analytical verification. Here we consider the geometry shown in Fig. 2 with $a = 3$ cm and $l = 10$ cm .In this simulation we have assumed that the potential $V(\rho, \phi\ )$ of the top side is 10V. Figures 3(a) to 3(c) show the electrostatic potential inside the cylinder in different distances from the z-axis, namely $\rho = 0, \rho = 1.5$ cm and $\rho = 2.5$ cm, using three different approaches. The results demonstrate an excellent agreement between the different approaches.

All problems considered thus far have been Laplace problems where there has been no charge density in the problem. The significance of deriving Green's functions is to tackle Poisson problems, in which the problem includes a charge density since in general Poisson problems are much harder to solve. To truly show the validity of our Green function we consider a Poisson problem. The geometry of this problem is illustrated in Fig. 4 with $a = 3$ cm and $l = 10$ cm. As shown in the figure, the top and bottom of the cylinder are grounded and the side of the cylinder is kept at 10 V. Additionally, we have included a total charge of $q = 2\pi r\lambda$ in the form of a ring of radius $r = 2$ cm and linear charge density $\lambda = 7.96 \times 10^{-11}\ \mathrm{C}/_{\mathrm{m}}$ placed at height $h = 5$ cm with azimuthal symmetry.

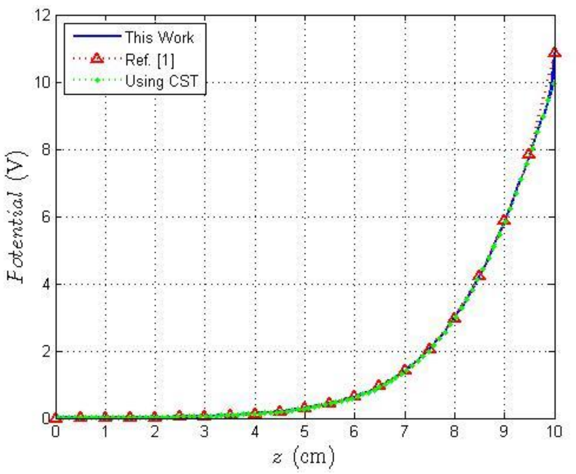

Fig. 3(a) Electrostatic potential inside the cylinder at $\rho = 0$ due to boundary conditions of Fig. 2.

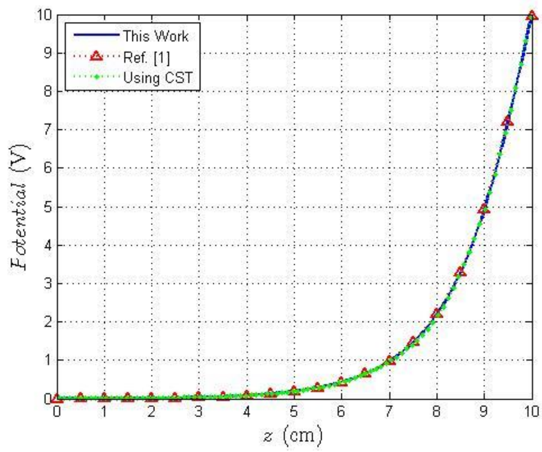

Fig. 3(b) Electrostatic potential inside the cylinder at $\rho = 1.5$ cm due to boundary conditions of Fig. 2.

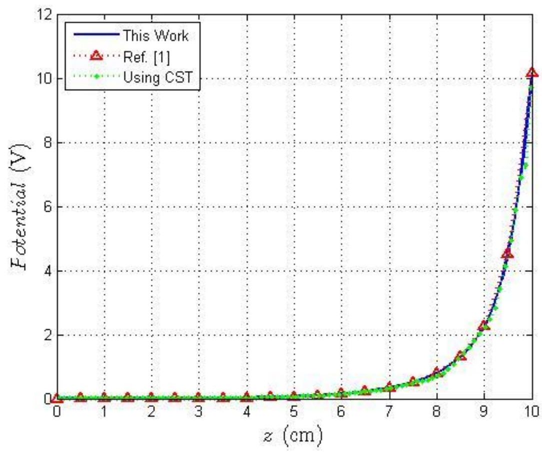

Fig. 3(c) Electrostatic potential inside the cylinder at $\rho = 2.5$ cm due to boundary conditions of Fig. 2.

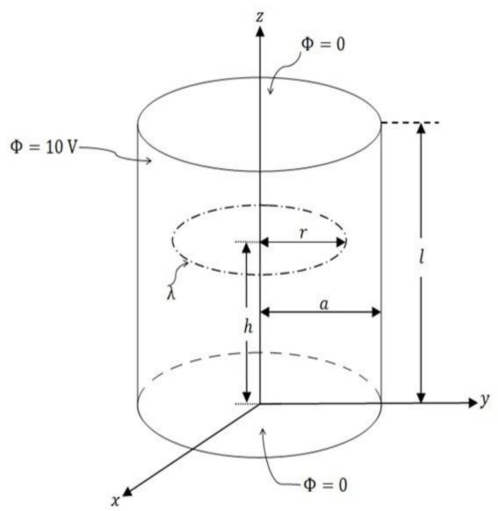

Fig. 4. Conducting cylinder of radius *a* and height *l*. There is a ring of radius *r* and linear charge density λ located at height *h* inside the cylinder.

Fig. 5(a) shows the numerical and the analytical values of the electrostatic potential along the axis of the cylinder. As it can be seen there is great agreement between the numerical and analytical results.

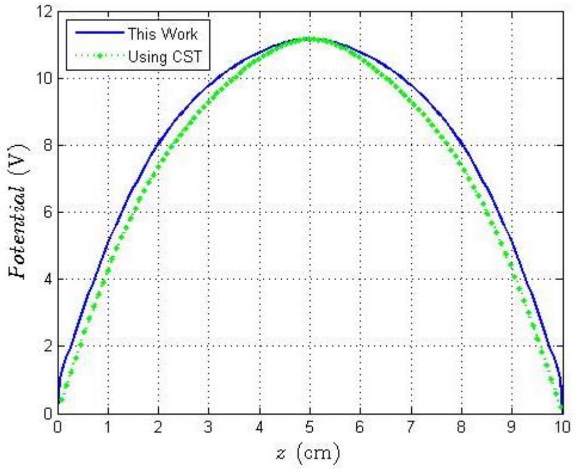

Fig. 5(a) Electrostatic potential inside the cylinder on the z axis due to the Poisson problem of Fig. 4.

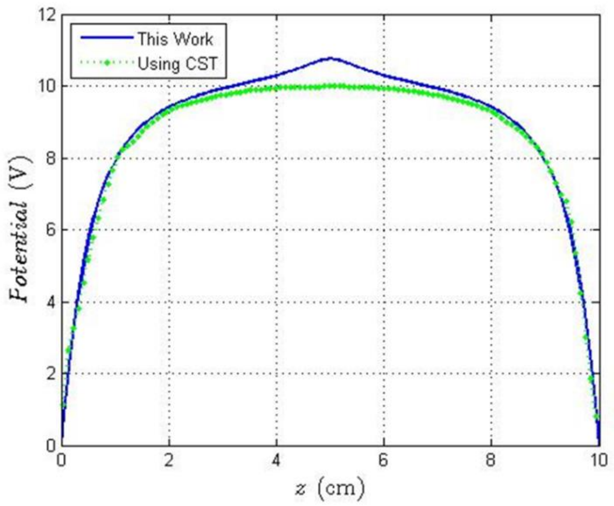

Fig. 5(b) Electrostatic potential inside the cylinder on $\boldsymbol{\rho = 1.5}$ cm due to the Poisson problem of Fig. 4.

Fig. 5(b) shows the numerical and the analytical values of the electrostatic potential at $\rho = 1.5$ cm for $0 \leq z \leq l$. As it can be seen, there is a small discrepancy between the numerical and the analytical results at $z = \frac{l}{2}$. Considering the geometry of the problem, one would expect the maximum potential at $z = \frac{l}{2}$ as shown by the analytical results. The inaccuracy of the numerical results can be attributed to the coarseness of volumetric meshing that was used due to computational limitations. More accurate numerical results can be obtained using a finer mesh.

6- ACKNOWLEDGEMENT

The authors would like to express their gratitude to Prof. F. Azadi Namin for the valuable contributions he made to the content of the paper.

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